Modular Markovian Logic

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Abstract. We introduce Modular Markovian Logic (MML) for compositional continuous-time and continuous-space Markov processes. MML combines operators specific to stochastic logics with operators reflecting the modular structure of the models, similar to those used by spatial and separation logics. We present a complete Hilbert-style axiomatization for MML, prove the small model property and analyze the relation between stochastic bisimulation and logical equivalence.

1 Introduction

Complex networks (e.g., embedded systems, communication networks, the Internet etc.) and complex systems (e.g., biological, ecological, social, financial, etc.) are often modelled as stochastic processes, to encapsulate a lack of knowledge or inherent randomness. Such systems are frequently modular in nature, consisting of parts which are systems in their own right. Their global behaviour depends on the behaviour of their parts and on the links which connect them. Understanding such systems requires integration of local stochastic information in a formal way, in order to address questions such as: *"to what extent is it possible to derive global properties of the system from the local properties of its modules?"*.

This is a problem of fundamental importance in complex systems that has been usually addressed semantically: probabilistic and stochastic process algebras, for instance, aim at describing compositionally the behaviour of a system from the behaviours of its subsystems taking into account various types of synchronization or communication. This approach is quite restrictive, as process algebras are not logics: one cannot express basic logical operations such as conjunction, disjunction, implication or negation of properties. Usually, to do this, people use logics such as temporal logics, modal μ-calculus [21] or Hennessy-Milner logic [18] to express properties of transition systems. But these are global properties only and no logical framework developed so far allows reasoning on stochastic systems and subsystems at the same time.

In this paper we develop a logical framework called Modular Markovian Logic (MML) that tackles this problem by organizing qualitative and quantitative properties of stochastic systems in hierarchical, modular structures, thereby proving global properties from the local properties of modules. Formally, denoting *"process P has the property ϕ"* by $P \Vdash \phi$ and letting "$\otimes$" be the composition operator, we aim to establish a framework containing modular proof rules of the form $\dfrac{P_1 \Vdash \phi_1, ..., P_k \Vdash \phi_k}{P_1 \otimes ... \otimes P_k \Vdash \rho}\, C(\rho, \phi_1, .., \phi_k)$, where C is a logical constraint.

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To gain this level of expressivity, MML combines stochastic operators similar to the ones of Aumann's system [1, 14] with modular operators similar to the ones used in spatial logics [5, 6] and in separation logics [30]. For an observable action a and a positive rational r, the operator "L_r^a" of MML expresses the fact that a process can perform an a-transition with the rate[1] at least r. In addition, the *composition operator* "|" joins logical terms and directly expresses properties of the combined subsystems.

On the semantic level, we introduce the *modular Markov processes* (MMPs) which are (continuous-) labelled Markov processes [13, 28] enriched with an algebraic structure. This algebra defines the composition of Markovian systems and establishes the relation between a system and its subsystems. The composition of behaviours satisfies a general synchronization pattern which subsumes most of the classical notions of parallel composition found in process algebras.

We define the modular Markovian logic for a semantics based on MMPs. We investigate the relation between stochastic bisimulations of MMPs and logical equivalence induced by MML over the class of MMPs. We present a complete Hilbert style axiomatization of MML for the Markovian semantics and prove the small model property.

Research context. Labelled Markov process (LMPs) are introduced in [12, 3, 13, 28] and they generalize most of the models of Markovian systems. A similar concept, Harsanyi type space (HTS), has been studied in the context of belief systems [15, 27]. MMPs are built on top of these, by exploiting their equivalence proved in [10]. In addition, MMPs have inbuilt an algebraic structure that extends, for continuous space and time, the concepts of the Markov chain algebra [4].

Probabilistic logics have been studied for LMPs (probabilistic versions of temporal and Hennessy-Milner logics [13, 11, 28]), for HTSs (Aumann's system [1, 14]) and in a more general context [9]. The first class focuses on model checking and logical characterization of stochastic bisimulation, while for Aumann's system also axiomatization issues have been addressed [17, 31]. In [8] we have proposed a completely axiomatized stochastic logic that combines features of the two classes of logics. In this paper we extend the stochastic logic with modular operators that allow us, in addition, to investigate the algebraic structures of the models.

Modular logics, such as spatial logics [5, 6] and separation logic [30] have been developed for concurrent nondeterministic systems, but to the best of our knowledge, no stochastic or probabilistic version of these have been studied.

The paper is organized as follows. Section 2 introduces basic concepts used in the paper. Section 3 defines MMPs and their bisimulation. Section 4 presents MML and results concerning the relationship between logical equivalence and bisimulation. Section 5 contains the axiomatic system of MML, the soundness and completeness metatheorems and the small model property. The paper also contains a conclusive section.

2 Preliminary definitions

In this section we establish the terminology used in the paper.

[1] The rate of a transition is the parameter of an exponentially distributed random variable that characterizes, for Markovian processes, the duration of the transition.

Given a set M, $\Sigma \subseteq 2^M$ that contains M and is closed under complement and countable union is a *σ-algebra* over M; (M, Σ) is a *measurable space* and the elements of Σ are *measurable sets*. $\Omega \subseteq 2^M$ is a *base for* Σ if Σ is the closure of Ω under complement and countable union; we write $\overline{\Omega} = \Sigma$.

A relation $\mathfrak{R} \subseteq M \times M$ is *non-wellfounded* if there exists $\{m_i \in M \mid i \in \mathbb{N}\}$ such that for each $i \in \mathbb{N}$, $(m_i, m_{i+1}) \in \mathfrak{R}$; otherwise it is *wellfounded*. A subset $N \subseteq M$ is $\mathfrak{R}$-closed iff $\{m \in M \mid \exists n \in N, (m, n) \in \mathfrak{R}\} \subseteq N$. If (M, Σ) is a measurable space and $\mathfrak{R} \subseteq M \times M$, $\Sigma(\mathfrak{R})$ denotes the set of measurable $\mathfrak{R}$-closed subsets of M.

A *measure* on (M, Σ) is a function $\mu : \Sigma \to \mathbb{R}^+$ such that $\mu(\emptyset) = 0$ and for $\{N_i | i \in I \subseteq \mathbb{N}\} \subseteq \Sigma$ with pairwise disjoint elements, $\mu(\bigcup_{i \in I} N_i) = \sum_{i \in I} \mu(N_i)$.

Let $\Delta(M, \Sigma)$ be the class of measures on (M, Σ). We organize it as a measurable space by considering the σ-algebra generated, for arbitrary $S \in \Sigma$ and $r > 0$, by the sets $\{\mu \in \Delta(M, \Sigma) : \mu(S) \geq r\}$.

Given two measurable spaces (M, Σ) and (N, Θ), a mapping $f : M \to N$ is *measurable* if for any $T \in \Theta$, $f^{-1}(T) \in \Sigma$. We use $[\![M \to N]\!]$ to denote the class of measurable mappings from (M, Σ) to (N, Θ).

Central for this paper is the notion of an *analytic set*. We only recall the main definition and mention the properties of analytic sets used in our proofs. For detailed discussion on this topic related to Markov processes, the reader is referred to [28] (Section 7.5) or to [10] (Section 4.4).

A metric space (M, d) is *complete* if every Cauchy sequence converges in M.

A *Polish space* is the topological space underlying a complete metric space with a countable dense subset. Note that any discrete space is Polish.

An *analytic set* is the image of a Polish space under a continuous function between Polish spaces. Note that any Polish space is an analytic set.

There are some basic facts about analytic sets that we use in this paper. Firstly, an analytic set, as measurable space, has a denumerable base with disjoint elements. Secondly, If $\mathcal{M}_1, \mathcal{M}_2$ are analytic sets with Σ_1, Σ_2 the Borel algebras generated by their topologies, then the product space $\mathcal{M} = \mathcal{M}_1 \times \mathcal{M}_2$ with the Borel algebra Σ generated by the product topology is an analytic set.

3 Modular Markov processes

For the beginning we introduce continuous Markov processes (CMPs) for a finite set $\mathcal{A}$ of *actions*. CMPs are coalgebraic structures that encode stochastic behaviors. If m is the current state of the system, N a measurable set of states and $a \in \mathcal{A}$, $\theta(a)(m)$ is a measure on the state space and $\theta(a)(m)(N) \in \mathbb{R}^+$ represents the *rate* of an exponentially distributed random variable that characterizes the duration of an a-transition from m to arbitrary $n \in N$. Indeterminacy is resolved by races between events executing at different rates.

Definition 1 (Continuous Markov processes). *Given an analytic set (M, Σ), where Σ is the Borel algebra generated by the topology, an $\mathcal{A}$-*continuous Markov kernel *is a*

tuple $\mathcal{K} = (M, \Sigma, \theta)$, where $\theta : \mathcal{A} \to [\![M \to \Delta(M, \Sigma)]\!]$.
If $m \in M$, $(\mathcal{K}, m)$ is an $\mathcal{A}$-continuous Markov process[2].

Let $\mathfrak{K}$ be the class of $\mathcal{A}$-CMKs; $\mathcal{K}, \mathcal{K}_i, \mathcal{K}'$ are used to range over $\mathcal{A}$-CMKs. Stochastic bisimulation follows the line of Larsen-Skou bisimulation [23, 11, 28].

Definition 2 (Stochastic Bisimilarity). *Given $\mathcal{K} = (M, \Sigma, \theta) \in \mathfrak{K}$, a* rate-bisimulation relation *on $\mathcal{K}$ is a relation $\mathfrak{R} \subseteq M \times M$ such that $(m, n) \in \mathfrak{R}$ iff for any $C \in \Sigma(\mathfrak{R})$ and any $a \in \mathcal{A}$, $\theta(a)(m)(C) = \theta(a)(n)(C)$.*
Two processes $(\mathcal{K}, m)$ and $(\mathcal{K}, n)$ are stochastic bisimilar*, written $m \sim_{\mathcal{K}} n$, if they are related by a rate-bisimulation relation.*

Two processes $(\mathcal{K}, m)$ and $(\mathcal{K}', m')$ are *stochastic bisimilar*, written $(\mathcal{K}, m) \sim (\mathcal{K}', m')$, iff $m \sim_{\mathcal{K} \uplus \mathcal{K}'} m'$, where $\mathcal{K} \uplus \mathcal{K}'$ is the disjoint union of $\mathcal{K}$ and $\mathcal{K}'$. We call the relation $\sim$ *stochastic bisimulation*.

3.1 Synchronization

To define the modular Markov processes we need a general notion of synchronization of CMPs that we introduce following the general line of [20]. For this, we assume extra structure on the set $\mathcal{A}$ of actions.

Firstly, we consider a *synchronisation function* $*$ that is a partial function $* : \mathcal{A} \times \mathcal{A} \hookrightarrow \mathcal{A}$ which associates to some $a, b \in \mathcal{A}$ an action $a * b \in \mathcal{A}$ interpreted as the synchronisation of a and b. In this way we can mimic various synchronisation paradigms. For instance, the CCS-style synchronisation [26] requires that $a * \overline{a} = \tau$, where $\tau \in \mathcal{A}$ is a special action; CSP-style [19] requires that $a * a = a$; for interleaving and ACP-style [2] we need to assume that there exists a reflexive transition $\delta \in \mathcal{A}$ such that for any $a \in \mathcal{A}$, $a * \delta = a$. Similarly, most classical notions of parallel composition in process algebras may be expressed by a suitable synchronization function.
The only formal requirement is that $*$, as an operation, is *commutative* ($a * b = b * a$).

Secondly, we assume a function $\bullet : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ that computes, given the rates r and s of the actions a and b respectively, the rate $r \bullet s$ of the synchronisation $a * b$. Examples of such function are the *mass action law* used with stochastic Pi-calculus [29] and other models of bio-chemical interactions and the *minimal rate law* used by PEPA [16] for applications in performance evaluation. The formal requirements are:

$\bullet : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ is a *continuous* function that, as an operation, is *commutative* ($r \bullet s = s \bullet r$), *associative* ($(r \bullet s) \bullet t = r \bullet (s \bullet t)$) and *bilinear* ($(r_1 + r_2) \bullet s = (r_1 \bullet s) + (r_2 \bullet s)$ and $s \bullet (r_1 + r_2) = (s \bullet r_1) + (s \bullet r_2)$).

These two functions define the synchronization of two CMPs as follows.

Definition 3. *For $i = 1, 2$, let $\mathcal{K}_i = (M_i, \Sigma_i, \theta_i) \in \mathfrak{K}$ and $\Delta_i \subseteq \Sigma_i$ denumerable bases with disjoint elements. $\mathcal{K} = (M, \Sigma, \theta)$ is the* product *of $\mathcal{K}_1$ and $\mathcal{K}_2$, written $\mathcal{K} = \mathcal{K}_1 \times \mathcal{K}_2$, if*

[2] $\theta(\alpha)$ is a measurable mapping between (M, Σ) and $\Delta(M, \Sigma)$. This is equivalent with the conditions on the two-variable *rate function* used in [13] to define continuous Markov processes (see, e.g. Proposition 2.9, of [10]).

$M = M_1 \times M_2$, $\Sigma = \overline{\Delta_1 \times \Delta_2}$ *and* $\theta : \mathcal{A} \to [M \to [\Sigma \to \mathbb{R}^+]]$ *is defined, for* $m_i \in M_i$, $a \in \mathcal{A}$ *and* $S = \bigcup_{k \in K \subseteq \mathbb{N}} U_k^1 \times U_k^2 \in \Sigma$ *for* $U_k^i \in \Delta_i$, *by*

$$\theta(a)((m_1, m_2))(S) = \sum_{(b,c)\in\mathcal{A}^2}^{b*c=a} \sum_{k \in K} \theta_1(b)(m_1)(U_k^1) \bullet \theta_2(c)(m_2)(U_k^2).$$

$\mathcal{K}$ represents the result of the synchronization of $\mathcal{K}_1$ and $\mathcal{K}_2$: θ calculates the rate of a by summing all the possible synchronizations $b * c = a$ between $\mathcal{K}_1$ and $\mathcal{K}_2$. The properties of $\bullet$ guarantee that the previous sum is convergent and independent of the choice of the bases. Because $\bullet$ is bilinear, $r \bullet 0 = 0$.

Lemma 1. *If* $\mathcal{K}_1, \mathcal{K}_2 \in \mathfrak{K}$, *then* $\mathcal{K}_1 \times \mathcal{K}_2 \in \mathfrak{K}$.

If $(\mathcal{K}_1, m_1)$ and $(\mathcal{K}_2, m_2)$ are CMPs, then $(\mathcal{K}_1 \times \mathcal{K}_2, (m_1, m_2))$ is a CMP called the *synchronization of* $(\mathcal{K}_1, m_1)$ and $(\mathcal{K}_2, m_2)$.

3.2 Parallel composition

For introducing a concept of parallel composition that is general enough to include most of the similar concepts, we assume that the support set of the Markov kernel has an algebraic structure called *modular structure*.

Definition 4 (Modular structure). *A tuple* $(M, \equiv, \otimes)$ *is a* modular structure *on a set* M *if* $\equiv \subseteq M \times M$ *is an equivalence relation and* $\otimes : M \times M \hookrightarrow M$ *is a partial operation which, with respect to* $\equiv$, *is*

– *a* **congruence***: if* $m_0 \equiv m_1$, *then* $m_0 \otimes m_2$ *is defined iff* $m_1 \otimes m_2$ *is defined and*
$$m_0 \otimes m_2 \equiv m_1 \otimes m_2,$$
– **associative***:* $(m_0 \otimes m_1) \otimes m_2$ *is defined iff* $m_0 \otimes (m_1 \otimes m_2)$ *is defined and*
$$(m_0 \otimes m_1) \otimes m_2 \equiv m_0 \otimes (m_1 \otimes m_2),$$
– **commutative***:* $m_0 \otimes m_1$ *is defined iff* $m_1 \otimes m_0$ *is defined and*
$$m_0 \otimes m_1 \equiv m_1 \otimes m_0,$$
– **modular***: if* $m_0 \otimes m_1 \equiv n_0 \otimes n_1$, *then either* $m_i \equiv n_j$ *and* $m_{1-i} \equiv n_{1-j}$ *for* $i, j \in \{0, 1\}$, *or there exist* $m_j^i \in M$ *for* $i, j \in \{0, 1\}$, *such that* $m_i \equiv m_0^i \otimes m_1^i$ *and* $n_i \equiv m_i^0 \otimes m_i^1$ *for* $i \in \{0, 1\}$*;*
– **wellfounded***: the relation* $\{(m, n) \mid \exists n' \in M, m \equiv n \otimes n'\}$ *is wellfounded.*

Process algebras are examples of modular structures where $\equiv$ is the structural congruence or some bisimulation relation, while $\otimes$ is, for instance, the parallel composition. In these cases well-foundedness expresses the fact that any process (modulo (Nil): $P \equiv P \otimes 0$) can be decomposed into a finite number of processes that cannot be, further, decomposed; and modularity guarantees the uniqueness of this decomposition up to structural congruence. In process algebras these hold, modulo (Nil), due to the inductive definition of the set of processes.

For modular structures, we lift the signature to sets by defining, for arbitrary $N, N' \subseteq M$, $N \otimes N' = \{m \in M \mid m \equiv n \otimes n'$ for some $n \in N, n' \in N'\}$. Moreover, if $\Sigma \subseteq 2^M$, let $\Sigma \otimes \Sigma = \{N \otimes N' \mid N, N' \in \Sigma\}$.

Definition 5 (Modular Markov process). *An* $\mathcal{A}$-modular Markov kernel *is a tuple* $\mathcal{M} = (\mathcal{K}, \equiv, \otimes)$*, where* $\mathcal{K} = (M, \Sigma, \theta) \in \mathfrak{K}$ *and* $(M, \equiv, \otimes)$ *is a modular structure such that its algebraic structure satisfy the following properties*
1. it preserves the Borel-algebras, i.e., $\Sigma \otimes \Sigma \subseteq \Sigma$*,*
2. it preserves the behaviours of modules and their synchronization, i.e.,
$(i).\ \equiv \subseteq \sim,\qquad (ii).\ (\mathcal{K}, m_0 \otimes m_1) \sim (\mathcal{K} \times \mathcal{K}, (m_0, m_1)).$
If $m \in M$*,* $(\mathcal{M}, m)$ *is a* modular Markov process.

Condition 2(ii) requires that $(\mathcal{K}, m_0 \otimes m_1)$ is bisimilar with the synchronization of $(\mathcal{K}, m_0)$ and $(\mathcal{K}, m_1)$.

M is called the *support of* $\mathcal{M}$, denoted $sup(\mathcal{M})$. Let $\mathfrak{M}$ be the class of $\mathcal{A}$-modular Markov kernels (MMKs); we use $\mathcal{M}, \mathcal{N}, \mathcal{M}_i, \mathcal{M}'$ to range over $\mathfrak{M}$.

Because MMKs preserves the synchronisation of the modules, stochastic bisimulation is a congruence.

Theorem 1 (Congruence). *Given* $(\mathcal{K}, \equiv, \otimes) \in \mathfrak{M}$*, if* $m \sim_{\mathcal{K}} m'$ *and both* $m \otimes n$ *and* $m' \otimes n$ *are defined, then* $m \otimes n \sim_{\mathcal{K}} m' \otimes n$*.*

4 Modular Markovian Logic

In this section we introduce Modular Markovian Logic (MML).

The formulas of MML are the elements of the set $\mathcal{L}$ introduced by the following grammar, for arbitrary $a \in \mathcal{A}$ and $r \in \mathbb{Q}_+$.

$$\phi := \top \ \vdots\ \neg\phi \ \vdots\ \phi \wedge \phi \ \vdots\ L_r^a \phi \ \vdots\ \phi | \phi.$$

The semantics is given by the *satisfiability relation* " $\Vdash$ " defined for $\mathcal{M} \in \mathfrak{M}$ and $m \in sup(\mathcal{M})$, inductively as follows.

$\mathcal{M}, m \Vdash \top$ always;
$\mathcal{M}, m \Vdash \neg\phi$ iff it is not the case that $\mathcal{M}, m \Vdash \phi$;
$\mathcal{M}, m \Vdash \phi \wedge \psi$ iff $\mathcal{M}, m \Vdash \phi$ and $\mathcal{M}, m \Vdash \psi$;
$\mathcal{M}, m \Vdash L_r^a \phi$ iff $\theta(a)(m)([\![\phi]\!]_{\mathcal{M}}) \geq r$, where $[\![\phi]\!]_{\mathcal{M}} = \{m \in M | \mathcal{M}, m \Vdash \phi\}$;
$\mathcal{M}, m \Vdash \phi_1 | \phi_2$ iff $m \equiv m_1 \otimes m_2$ and $\mathcal{M}, m_i \Vdash \phi_i$, $1 = 1, 2$.

"|" is a polyadic modality of arity 2. The formula $L_r^a \phi$ is interpreted as *"the rate of an a-transition from the current state to a state satisfying* ϕ *is at least r"*. Notice that the semantics of $L_r^a \phi$ is well defined only if $[\![\phi]\!]_{\mathcal{M}}$ is measurable. This is guaranteed by the next lemma.

Lemma 2. *For any* $\phi \in \mathcal{L}$ *and any* $\mathcal{M} = (M, \Sigma, \theta) \in \mathfrak{M}$*,* $[\![\phi]\!]_{\mathcal{M}} \in \Sigma$*.*

When it is not the case that $\mathcal{M}, m \Vdash \phi$, we write $\mathcal{M}, m \nVdash \phi$. A formula ϕ is *satisfiable* if there exists $\mathcal{M} \in \mathfrak{M}$ and $m \in sup(\mathcal{M})$ such that $\mathcal{M}, m \Vdash \phi$. If $\neg\phi$ is not satisfiable, ϕ is *valid*, denoted by $\Vdash \phi$.

In what follows we consider all the Boolean derived operators. In addition, let $\boxed{+}_{i=1..n} \phi_i = \bigwedge_{i,j=1..n}^{i \neq j} (\phi_i \rightarrow \neg\phi_j)$ and $\bot = \neg\top$ and for $k \in \mathbb{N}$, let $k = \neg(\underbrace{\top | \top | .. | \top}_{k+1})$; notice that $\mathcal{M}, m \Vdash k$ iff m can be decomposed in maximum k modules.

In the rest of this section we focus on the logical equivalence induced by MML on MMPs and its relation to stochastic bisimulation on MMPs. The next theorem states that $\equiv$ preserves the satisfiability of $\mathcal{L}$ formulas.

Theorem 2. *For $\mathcal{M} \in \mathfrak{M}$ and $m, n \in \sup(\mathcal{M})$, if $m \equiv n$, then for all $\phi \in \mathcal{L}$, $\mathcal{M}, m \Vdash \phi$ iff $\mathcal{M}, n \Vdash \phi$.*

Let $\mathcal{L}^* \subsetneq \mathcal{L}$ be defined by the grammar $\phi := \top \vdots \neg\phi \vdots \phi \wedge \phi \vdots L_r^a\phi$. The next theorem reproduces a similar result presented in [13, 28].

Theorem 3. *Given $\mathcal{M} \in \mathfrak{M}$ and $m, n \in \sup(\mathcal{M})$, if [for any $\phi \in \mathcal{L}^*$, $\mathcal{M}, m \Vdash \phi$ iff $\mathcal{M}, n \Vdash \phi$], then $m \sim n$.*

5 A complete Hilbert-style axiomatization for MML

Tables 1, 2 and 3 contain a Hilbert-style axiomatization for MML.

The stochastic axioms in Table 1 have been proposed in [8] where we have proved that they form a complete axiomatization for CMPs. These axioms are similar, but more complex due to stochasticity, than the ones proposed in [31] for Harsanyi type spaces. As in the probabilistic case, we have infinitary rules (R2) and (R3) that encode the Archimedean properties of $\mathbb{Q}$. However, a finitary axiomatization is possible on the lines of [17] at the price of defining some complex operators, as shown in [22].

Table 1: Stochastic Axioms

(A1): $\vdash L_0^a\phi$
(A2): $\vdash L_{r+s}^a\phi \to L_r^a\phi$
(A3): $\vdash L_r^a(\phi \wedge \psi) \wedge L_s^a(\phi \wedge \neg\psi) \to L_{r+s}^a\phi$
(A4): $\vdash \neg L_r^a(\phi \wedge \psi) \wedge \neg L_s^a(\phi \wedge \neg\psi) \to \neg L_{r+s}^a\phi$
(R1): If $\vdash \phi \to \psi$ then $\vdash L_r^a\phi \to L_r^a\psi$
(R2): If $\forall r < s, \vdash \phi \to L_r^a\psi$ then $\vdash \phi \to L_s^a\psi$
(R3): If $\forall r > s, \vdash \phi \to L_r^a\psi$ then $\vdash \phi \to \bot$

Table 2: Structural Axioms

(A5): $\vdash (\phi|\psi)|\rho \to \phi|(\psi|\rho)$
(A6): $\vdash \phi|\psi \to \psi|\phi$
(A7): $\vdash \phi|\bot \to \bot$
(A8): $\vdash \phi|(\psi \vee \rho) \to (\phi|\psi \vee \phi|\rho)$
(R4): If $\vdash \phi \to \psi$ then $\vdash \phi|\rho \to \psi|\rho$
(R5): If $\vdash \phi \to \phi|\top$ then $\vdash \phi \to \bot$

The structural axioms in Table 2 are similar to the axioms proposed in [25] for a spatial logic on CCS semantics. The main difference is rule (R5) which rejects models that do not respect the modularity conditions. An example is the rule (Nil): $P \equiv P|0$ which allows processes with (trivial) non-wellfounded structure. However, one can easily make an MMP from a process algebra term by simply taking the quotient of the class of processes by (Nil) and similar rules.

To simplify the form of the modular axioms in Table 3, we use some additional notations. π_k is the set of permutations of $\{1, .., k\}$. For arbitrary $a \in \mathcal{A}$, we assume that the set of its $*$-decompositions (which is finite) is indexed and let $I_a = \{i \mid b_i * c_i = a\}$. If $\{(r_k^{i,j}) \mid i \in I, k \in K, j \in \{0,1\}\}, \{(s_k^{i,j}) \mid i \in I, k \in K, j \in \{0,1\}\} \subseteq \mathbb{Q}_+$, let $r_K^I \bullet s_K^I = \sum_{i \in I} \sum_{j \in \{0,1\}}^{k \in K} (r_k^{i,j}) \bullet (s_k^{i,j})$.

The rules (R6) and (R7) encode the well-foundedness and the modularity of the models. The rules (R8) and (R9) are logical versions of classical *expansion laws* for parallel operator. (R8) states that the rate of the a-transitions from m to $[\![\rho]\!]$ is at least the sum, after $k \in K$, of all $\bullet$-products of the rates of b and c-transitions (for $a = b * c$) from m_1 and m_2 to $[\![\phi_k^j]\!]$ and $[\![\phi_k^{1-j}]\!]$ respectively ($j = 0, 1$), given that $m \equiv m_1 \otimes m_2$ and $[\![\rho]\!]$ covers $\bigcup_{k \in K} [\![\phi_k^j]\!] \otimes [\![\phi_k^{1-j}]\!]$. For instance, $\vdash (L_r^b \phi \wedge L_u^c \psi)|(L_s^c \psi \wedge L_v^b \phi) \to L^{b*c}_{(r \bullet s)+(u \bullet v)} \phi|\psi$ and $\vdash L_r^b \top | L_s^c \top \to L_{r \bullet s}^{b*c} \top$ are instances of (R8). Similarly, (R9) states that the rate of the a-transitions from m to $[\![\rho]\!]$ is strictly bigger than the sum, after $k \in K$, of all $\bullet$-products of the rates of b and c-transitions (for $a = b * c$) from m_1 and m_2 to $[\![\phi_k^j]\!]$ and $[\![\phi_k^{1-j}]\!]$ respectively ($j = 0, 1$), given that $m \equiv m_1 \otimes m_2$ and $[\![\rho]\!]$ is covered by $\bigcup_{k \in K} [\![\phi_k^j]\!] \otimes [\![\phi_k^{1-j}]\!]$. $\vdash (\neg L_r^b \top \wedge \neg L_u^c \top)|(\neg L_s^c \top \wedge \neg L_v^b \top) \to \neg L^a_{(r \bullet s)+(u \bullet v)}(\top|\top)$ is an instance of (R9) given that b, c are the only actions such that $a = b * c$.

Table 3: Modular axioms

(R6): If I is finite and $\vdash 1 \to \bigvee_{i \in I} \phi_i$, then $\vdash k \to \bigvee_{i_j \in I}^{s<k} \phi_{i_1}|...|\phi_{i_s}$.

(R7): If $\vdash \bigwedge_{i=1..k}^{j=0,1} (\phi_i^j \to 1)$ and $\vdash \phi_1^0|..|\phi_k^0 \to \phi_1^1|..|\phi_l^1$, then $k = l$ and

$$\vdash \bigvee_{\sigma \in \pi_k} \bigwedge_{i=1..k} \phi_i^0 \leftrightarrow \phi_{\sigma(i)}^1$$

(R8): If K is finite and $\vdash (\boxed{+}_{k \in K} \phi_k^0) \wedge (\boxed{+}_{k \in K} \phi_k^1) \wedge (\bigvee_{k \in K} \phi_k^0|\phi_k^1 \to \rho)$, then

$$\vdash \left(\bigwedge_{k \in K}^{i \in I_a} \bigwedge_{j=0,1} L^{b_i}_{(r_k^{i,j})} \phi_k^j \right) \Big| \left(\bigwedge_{k \in K}^{i \in I_a} \bigwedge_{j=0,1} L^{c_i}_{(s_k^{i,j})} \phi_k^{1-j} \right) \to L^a_{(r_K^I \bullet s_K^I)} \rho$$

(R9): If K is finite and $\vdash (\boxed{+}_{k \in K} \phi_k^0) \wedge (\boxed{+}_{k \in K} \phi_k^1) \wedge (\rho \to \bigvee_{k \in K} \phi_k^0|\phi_k^1)$, then

$$\vdash \left(\bigwedge_{k \in K}^{i \in I_a} \bigwedge_{j=0,1} \neg L^{b_i}_{(r_k^{i,j})} \phi_k^j \right) \Big| \left(\bigwedge_{k \in K}^{i \in I_a} \bigwedge_{j=0,1} \neg L^{c_i}_{(s_k^{i,j})} \phi_k^{1-j} \right) \to \neg L^a_{(r_K^I \bullet s_K^I)} \rho$$

As usual, we say that a formula ϕ is *provable*, denoted by $\vdash \phi$, if it can be proved from the axioms (using also Boolean rules). ϕ is *consistent*, if $\phi \to \bot$ is not provable. Given a set $\Phi, \Psi \subseteq \mathcal{L}$, Φ proves Ψ if from the formulas of Φ and the axioms we can prove all $\psi \in \Psi$; we write $\Phi \vdash \Psi$. Φ is consistent if it is not the case that $\Phi \vdash \bot$. For a sublanguage $L \subseteq \mathcal{L}$, we call Φ *L-maximally consistent* if Φ is consistent and no formula of L can be added to it without making it inconsistent. For $\Lambda_1, \Lambda_2 \subseteq \mathcal{L}$, $\Lambda_1|\Lambda_2 = \{\phi_1|\phi_2 : \phi_i \in \Lambda_i, i = 1, 2\}$.

Theorem 4 (Soundness). *The axiomatic system of MML is sound for the Markovian semantics, i.e., for any $\phi \in \mathcal{L}$, if $\vdash \phi$ then $\Vdash \phi$.*

In what follows we prove the finite model property for MML by constructing a model for a given consistent formula. This result will eventually prove that the axiomatic system is also complete for the Markovian semantics, meaning that everything that is true for all the models can be proved. Before proceeding, we fix some notations.

For $n \in \mathbb{N}$, $n \neq 0$, let $\mathbb{Q}_n = \{\frac{p}{n} : p \in \mathbb{N}\}$. If $S \subseteq \mathbb{Q}$ is finite, the *granularity of* S, $gr(S)$, is the lowest common denominator of the elements of S.
The *modal depth of* $\phi \in \mathcal{L}$ is defined by $md(\top) = 0$, $md(\neg\phi) = md(\phi)$, $md(L^a_r\phi) = md(\phi) + 1$ and $md(\phi \wedge \psi) = md(\phi|\psi) = max(md(\phi), md(\psi))$.
The *structural depth of* $\phi \in \mathcal{L}$ is defined by $sd(\neg\phi) = sd(L^a_r\phi) = sd(\phi)$, $sd(\phi \wedge \psi) = max(sd(\phi), sd(\psi))$ and $sd(\phi|\psi) = sd(\phi) + sd(\psi) + 1$.
The *granularity of* $\phi \in \mathcal{L}$ is $gr(\phi) = gr(R)$, where $R \subseteq \mathbb{Q}_+$ is the set of indexes r of the operators L^a_r present in ϕ; the *upper bound of* ϕ is $max(\phi) = max(R)$.
For arbitrary $n \in \mathbb{N}$, let $\mathcal{L}_n$ be the sublanguage of $\mathcal{L}$ that uses only modal operators L^a_r with $r \in \mathbb{Q}_n$. For $\Lambda \subseteq \mathcal{L}$, let $[\Lambda]_n = \Lambda \cup \{\phi \in \mathcal{L}_n : \Lambda \vdash \phi\}$.

Consider a consistent formula $\psi \in \mathcal{L}$ with $gr(\psi) = n$ and $sd(\psi) = e$.
Let $\mathcal{L}[\psi] = \{\phi \in \mathcal{L}_n \mid max(\phi) \leq max(\psi), md(\phi) \leq md(\psi)$ and $sd(\phi) \leq sd(\psi)\}$.

In what follows we construct $\mathcal{M}_\psi \in \mathfrak{M}$ such that each $\Gamma \in \sup(\mathcal{M}_\psi)$ is a consistent set of formulas that contains an $\mathcal{L}[\psi]$-maximally consistent set and each $\mathcal{L}[\psi]$-maximally consistent set is contained in some $\Gamma \in \sup(\mathcal{M}_\psi)$. And we prove the truth lemma stating that for any $\phi \in \mathcal{L}[\psi]$, $\phi \in \Gamma$ iff $\mathcal{M}_\psi, \Gamma \Vdash \phi$.

Let $\Omega[\psi]$ be the set of $\mathcal{L}[\psi]$-maximally consistent sets of formulas. $\Omega[\psi]$ is finite and any $\Lambda \in \Omega[\psi]$ contains finitely many nontrivial formulas[3]; in the rest of this construction we only count non-trivial formulas while ignoring the rest.

For each $\Lambda \in \Omega[\psi]$, such that $\{\phi_1, ..., \phi_i\}$ is the set of its non-trivial formulas, we construct $\Lambda^+ \supseteq [\Lambda]_n$ with the property that $\forall\phi \in \Lambda$ and $a \in \mathcal{A}$ there exists $\neg L^a_r\phi \in \Lambda^+$.

The step [ϕ_1 and Λ:] (R3) guarantees that $\exists r \in \mathbb{Q}_n$ s.t. $[\Lambda]_n \cup \{\neg L^a_r\phi_1\}$ is consistent. Let $y^a_1 = min\{s \in \mathbb{Q}_n : [\Lambda]_n \cup \{\neg L^a_s\phi_1\}$ is consistent$\}$ and $x^a_1 = max\{s \in \mathbb{Q}_n : L^a_s\phi_1 \in [\Lambda]_n\}$ ((R3) guarantees the existence of max). (R2) implies that $\exists r \in \mathbb{Q} \setminus \mathbb{Q}_n$ s.t., $x^a_1 < r < y^a_1$ and $\{\neg L^a_r\phi_1\} \cup [\Lambda]_n$ is consistent. Let $n_1 = gran\{1/n, r\}$. Let $s^a_1 = min\{s \in \mathbb{Q}_{n_1} : [\Lambda]_{n_1} \cup \{\neg L^a_s\phi_1\}$ is consistent$\}$, $\Lambda^a_1 = \Lambda \cup \{\neg L^a_{s_1}\phi_1\}$ and $\Lambda_1 = \bigcup_{a \in A} \Lambda^a_1$.

We repeat this step of the construction for [ϕ_2 and Λ_1],..,[ϕ_i and Λ_{i-1}] and we obtain $\Lambda \subseteq \Lambda_1 \subseteq ... \subseteq \Lambda_i$, where Λ_i is a consistent set containing a finite set of nontrivial formulas. Let $n_\Lambda = gran\{1/n_1, .., 1/n_i\}$. We make this construction for all $\Lambda \in \Omega[\psi]$. Let $v = gran\{1/n_\Lambda : \Lambda \in \Omega[\psi]\}$. Let $\Lambda^+ = [\Lambda_i]_v$ and $\Omega^+[\psi] = \{\Lambda^+ : \Lambda \in \Omega[\psi]\}$. Notice that $v > n$; we call v the *parameter of* $\Omega[\psi]$.

Remark 1. For each $\Lambda \in \Omega[\psi]$, $\phi \in \Lambda$ and $a \in \mathcal{A}$, there exist $s, t \in \mathbb{Q}_v$, $s < t$, such that $L^a_s\phi, \neg L^a_r\phi \in \Gamma^+$. Moreover, there exists $f \in \Lambda^+$ such that $f \vdash \Lambda^+$.

Let Ω_v be the set of $\mathcal{L}_v$-maximally consistent sets of formulas and $\sigma : \Omega^+[\psi] \to \Omega_v$ be an injection such that for any $\Lambda^+ \in \Omega^+[\psi]$, $\Lambda^+ \subseteq \sigma(\Lambda^+)$. Let $\Omega_v[\psi] = \sigma(\Omega^+[\psi])$, and for $\phi \in \mathcal{L}[\psi]$, let $[\![\phi]\!] = \{\Gamma \in \Omega_v[\psi] : \phi \in \Gamma\}$.

Lemma 3. *(1) $\Omega_v[\psi]$ is finite.* *(2)* $2^{\Omega_v[\psi]} = \{[\![\phi]\!] \mid \phi \in \mathcal{L}[\psi]\}$.
(3) For any $\phi_1, \phi_2 \in \mathcal{L}[\psi]$, $\vdash \phi_1 \to \phi_2$ iff $[\![\phi_1]\!] \subseteq [\![\phi_2]\!]$.
(4) For any $\Gamma \in \Omega_v[\psi]$, $\phi \in \mathcal{L}[\psi]$ and $a \in \mathcal{A}$, there exist
$x = max\{r \in \mathbb{Q}_v : L^a_r\phi \in \Gamma\}$, $y = min\{r \in \mathbb{Q}_v : \neg L^a_r\phi \in \Gamma\}$ *and* $y = x + 1/v$.

[3] By nontrivial formulas we mean the formulas that are not obtained from more basic consistent ones by boolean derivations.

Let Ω be the set of $\mathcal{L}$-maximally consistent sets of formulas and $\pi : \Omega_v \to \Omega$ an injection such that for any $\Gamma \in \Omega_v$, $\Gamma \subseteq \pi(\Gamma)$.

Lemma 4. *For any $\Gamma \in \Omega_v[\psi]$, any $\phi \in \mathcal{L}[\psi]$ and any $a \in \mathcal{A}$, there exist* $x^\infty = sup\{r \in \mathbb{Q} : L_r^a\phi \in \pi(\Gamma)\} = inf\{r \in \mathbb{Q} : \neg L_r^a\phi \in \pi(\Gamma)\}$ *and* $x \leq x^\infty < y$.

We denote by $a_\phi^\Gamma = x^\infty$ defined for $\phi \in \mathcal{L}[\psi]$, $\Gamma \in \Omega_v[\psi]$ and $a \in \mathcal{A}$.

Lemma 5. $\mathcal{M}_\psi = (\mathcal{K}_\psi, \equiv, \otimes) \in \mathfrak{M}$*, where* $\mathcal{K}_\psi = (\Omega_v[\psi], 2^{\Omega_v[\psi]}, \theta_\psi)$ *with*
(i). θ_ψ *defined for arbitrary* $\phi \in \mathcal{L}[\psi]$, $\Gamma \in \Omega_v[\psi]$, $a \in \mathcal{A}$ *by* $\theta_\psi(a)(\Gamma)([\![\phi]\!]) = a_\phi^\Gamma$,
(ii). $\otimes$ *and* $\equiv$ *implicitly defined for* $\Gamma, \Gamma', \Gamma'' \in \Omega_v[\psi]$ *by* $[\Gamma \equiv \Gamma' \otimes \Gamma''$ *iff* $\Gamma'|\Gamma'' \subseteq \Gamma]$.

Proof. This proof is rather complex. we only sketch here the main arguments. Lemma 3(3) proves that for arbitrary $\Gamma \in \Omega_v[\psi]$ and $a \in \mathcal{A}$, $\theta_\psi(a)(\Gamma)$ is well defined.

To prove that $\theta_\psi(a)(\Gamma)$ is a measure, we show that $\theta_\psi(a)(\Gamma)([\![\bot]\!]) = 0$ and that for $\phi_1, \phi_2 \in \mathcal{L}[\psi]$ with $\vdash \phi_1 \to \neg\phi_2$, $\theta_\psi(a)(\Gamma)([\![\phi_1]\!]) + \theta_\psi(a)(\Gamma)([\![\phi_2]\!]) = \theta_\psi(a)(\Gamma)([\![\phi_1 \vee \phi_2]\!])$. These use the stochastic and the structural axioms, especially the archimedian rules.

The modular structure of $\mathcal{M}_\psi$ is proved based on Rules (R5), (R6) and (R7).

It remains to prove that $(\mathcal{M}_\psi, \Gamma' \otimes \Gamma'') \sim (\mathcal{M}_\psi \times \mathcal{M}_\psi, (\Gamma', \Gamma''))$. This requires to prove that $\Gamma'|\Gamma'' \subseteq \Gamma$ implies that for arbitrary $\phi \in \mathcal{L}[\psi]$,

$$\theta_\psi(a)(\Gamma)([\![\phi]\!]) = \sum_{b*c=a} \sum_{[\![\phi]\!]=[\![g'|g'']\!]}^{g',g'' \in \mathcal{F}^*} \theta_\psi(b)(\Gamma')([\![g']\!]) \bullet \theta_\psi(c)(\Gamma'')([\![g'']\!]).$$

This prove is done by involving the Rules (R8) and (R9) that approximates, from below and from above the value of $\theta_\psi(a)(\Gamma)([\![\phi]\!])$. Also here the archimedian rules play a central role together with the hypothesis of the continuity of $\bullet$.

Now we can prove the Truth Lemma.

Lemma 6 (Truth Lemma). *If* $\phi \in \mathcal{L}[\psi]$ *and* $\Gamma \in \Omega_v[\psi]$*, then* $[\mathcal{M}_\psi, \Gamma \Vdash \phi$ *iff* $\phi \in \Gamma]$.

Proof. Induction on the structure of ϕ. The Boolean cases are trivial.
The case $\phi = L_r^a\phi'$**:** $(\Longrightarrow)$ Suppose that $\mathcal{M}_\psi, \Gamma \Vdash L_r^a\phi'$ and $L_r^a\phi' \notin \Gamma$. Because Γ is $\mathcal{L}[\psi]$-maximally consistent, $\neg L_r^a\phi' \in \Gamma$. Let $y = min\{r \in \mathbb{Q}_p : \neg L_r^a\phi' \in \Gamma\}$. Then, from $\neg L_r^a\phi' \in \Gamma$, we obtain $r \geq y$. But $\mathcal{M}_\psi, \Gamma \Vdash L_r^a\phi'$ is equivalent with $\theta_\psi(a)(\Gamma)([\![\phi']\!]) \geq r$, i.e. $a_{\phi'}^\Gamma \geq r$. On the other hand, in Lemma 4 we proved that $a_{\phi'}^\Gamma < y$ - contradiction.
$(\Longleftarrow)$ Suppose that $L_r^a\phi' \in \Gamma$. Then $r \leq a_\phi^\Gamma$, implying $\theta_\psi(a)(\Gamma)([\![\phi]\!]) \geq r$.
The case $\phi = \phi_1|\phi_2$**:** $(\Longrightarrow)$ If $\mathcal{M}_\psi, \Gamma \Vdash \phi_1|\phi_2$, then $\Gamma = \Gamma_1 \otimes \Gamma_2$ and $\mathcal{M}_\psi, \Gamma_i \Vdash \phi_i$, $i = 1, 2$. The inductive hypothesis implies that $\phi_i \in \Gamma_i$ and because $\Gamma_1|\Gamma_2 \subseteq \Gamma$, $\phi_1|\phi_2 \in \Gamma$.
$(\Longleftarrow)$ If $\phi_1|\phi_2 \in \Gamma$, then there exist Γ_i with $\phi_i \in \Gamma_i$ and $\Gamma_1|\Gamma_2 \subseteq \Gamma$, i.e. $\Gamma = \Gamma_1 \otimes \Gamma_2$.

The previous lemma implies the small model property for our logic.

Theorem 5 (Small model property). *For any consistent formula* ϕ*, there exists* $\mathcal{M} \in \mathfrak{M}$ *with the cardinality of* $sup(\mathcal{M})$ *bound by the structure of* ϕ*, and* $m \in sup(\mathcal{M})$ *such that* $\mathcal{M}, m \Vdash \phi$.

The small model property proves the completeness of the axiomatic system.

Theorem 6 (Completeness). *MML is complete with respect to the Markovian semantics, i.e. if* $\Vdash \psi$*, then* $\vdash \psi$*.*

Proof. The proof is based on the fact that any consistent formula has a model. Indeed, [$\Vdash \psi$ implies $\vdash \psi$] is equivalent with [$\nvdash \psi$ implies $\nVdash \psi$], that is equivalent with [the consistency of $\neg\psi$ implies that there exists a model $\mathcal{M}$ such that $\mathcal{M}, m \nVdash \psi$], that is equivalent with [the consistency of $\neg\psi$ implies the satisfiability of $\neg\psi$].

6 Conclusions and future work

In this paper we have introduced Modular Markovian Logic, a new logic that combines features of stochastic and modular logics. Its semantics is in terms of modular Markov processes which are compositional continuous-time and continuous-space Markov processes. MML is appropriate for specifying and verifying modular properties of stochastic systems and to prove global properties from local properties of subsystems. For instance modular proof rules as the ones below can be given as instances of (R9).

$$\frac{P \Vdash L_r^b\top,\quad P' \Vdash L_s^c\top}{P\otimes P' \Vdash L_{r\bullet s}^{b*c}\top} \quad\text{and}\quad \frac{P \Vdash L_r^b\phi \wedge L_u^c\psi,\quad P' \Vdash L_s^c\psi \wedge L_v^b\phi}{P\otimes P' \Vdash L_{(r\bullet s)+(u\bullet v)}^{b*c}\rho}\ (\vdash \phi|\psi \to \rho).$$

Similarly, if b, c are unique such that $a = b*c$ and P, P' are unique such that $P'' \equiv P\otimes P'$, the rule below is based on an instance of (R10).

$$\frac{P \Vdash \neg L_r^b\top \wedge \neg L_u^c\top,\quad P' \Vdash \neg L_s^c\top \wedge \neg L_v^b\top}{P'' \Vdash \neg L_{(r\bullet s)+(u\bullet v)}^{a}(\top|\top)}$$

In this paper we have presented a complete Hilbert-style axiomatization for MML and prove the small model property. For future work we intend to focus on decidability and complexity problems following the line of [24], as well as on axiomatizations of model checking and possible procedures to automatize the proof of modular rules.

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